

ARITHMETIC PROGRESSIONS AND ITS APPLICATIONS TO (m, q) -ISOMETRIES: A SURVEY

TERESA BERMÚDEZ, ANTONIO MARTINÓN, AND JUAN AGUSTÍN NODA
UNIVERSITY OF LA LAGUNA

ABSTRACT. In this paper we collect some results about arithmetic progressions of higher order, also called polynomial sequences. Those results are applied to (m, q) -isometric maps.

CONTENTS

1. INTRODUCTION	2
2. SEQUENCES IN GROUPS	4
2.1. The difference operator	5
2.2. Some combinatorial results	5
2.3. The general term of a sequence	8
3. ARITHMETIC PROGRESSIONS	9
3.1. The term general of an arithmetic progression	9
3.2. Subsequences of an arithmetic progression	12
3.3. Sequences of arithmetic progressions	14
3.4. Perturbation of arithmetic progressions by nilpotents on a ring	15
4. NUMERICAL ARITHMETIC PROGRESSIONS	18
4.1. Recursive equations	18
4.2. Subsequences of numerical sequences	19
4.3. Monotony of positive arithmetic progressions	20
4.4. Powers of positive sequences	20
5. APPLICATIONS TO (m, q) -ISOMETRIES	24
5.1. (m, q) -isometries on metric spaces	24

Date: September 4, 2014.

2010 Mathematics Subject Classification. 40-02, 47-02, 47B99, 54E40.

Key words and phrases. (m, q) -isometry, m -isometry, n -nilpotent operator, arithmetic progression.

5.2.	Distance associated to an (m, q) -isometry on a metric space	26
5.3.	Powers and products of (m, q) -isometries on metric spaces	27
5.4.	(m, q) -isometries on Banach spaces	28
5.5.	m -isometries on Hilbert spaces	29
5.6.	A perturbation result on Hilbert spaces	30
5.7.	n -invertible operators on Hilbert spaces	30
	References	31

1. INTRODUCTION

In this paper we collect some results about arithmetic progressions of higher order, also called polynomial sequences. The papers of J. Alonso [6] and V. Dlab [17] are dedicated to this topic.

We consider arithmetic progressions on commutative groups. A sequence $(a_n)_{n \geq 0}$ in a group G is an *arithmetic progression of order h* if

$$\sum_{k=0}^{h+1} (-1)^{h+1-k} \binom{h}{k} a_{n+k} = 0 ,$$

for any integer $n \geq 0$; equivalently, there exists a polynomial p_a , with coefficients in G of degree less or equal to h , such that $p_a(n) = a_n$, for any $n \geq 0$; that is, there are $\gamma_h, \gamma_{h-1}, \dots, \gamma_1, \gamma_0$ in G such that

$$a_n = \gamma_h n^h + \gamma_{h-1} n^{h-1} + \dots + \gamma_2 n^2 + \gamma_1 n + \gamma_0 ,$$

for any $n \geq 0$. An arithmetic progression of order h is of *strict order h* if $h = 0$ or if $h > 1$ and it is not of order $h - 1$.

We pay attention in certain aspects of the theory of arithmetic progressions which are related with (m, q) -isometric maps.

J. Agler [2] introduced the notion of m -isometry for a positive integer m : an operator T acting on a Hilbert space H is an *m -isometry* if

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} T^{*k} T^k = 0 ,$$

where T^* denotes the adjoint operator of T . In [3], [4] and [5] the m -isometries are intensively studied.

The above definition of m -isometry is equivalent to that $(T^{*n} T^n)_{n \geq 0}$ is an arithmetic progression of order $m - 1$ in the algebra $L(H)$ of all (bounded linear) operators on H . Several authors have extended the concept

of m -isometry to the setting of Banach spaces. For more details see [7], [16], [19] and [21]. In [10] it was extended to metric spaces.

Now we summarize the content of this paper.

Section 2. We consider sequences in the setting of groups. First we study the *difference operator* D which acts on a sequence $a = (a_n)_{n \geq 0}$ in G in the following way:

$$Da = (a_1 - a_0, a_2 - a_1, a_3 - a_2, \dots).$$

The expression of the general term of the sequence $D^h a$, where D^h is the h power of D is given by

$$D^h a_n = \sum_{k=0}^h (-1)^{h-k} \binom{h}{k} a_{n+k}.$$

We collect some combinatorial results which are necessary and obtain an expression of the general term of any sequence a : for $n = 0, 1, 2, 3, \dots$,

$$a_n = \sum_{k=0}^n \binom{n}{k} D^k a_0.$$

Section 3. If a is an arithmetic progression of order h , then we obtain that

$$a_n = \sum_{k=0}^h \binom{n}{k} D^k a_0.$$

Other expressions for a_n are obtained.

We consider a double sequence $(a_{i,j})_{i,j \geq 0}$ such that its files and columns are arithmetic progressions of order k and h , respectively. Then we obtain that the diagonal sequence $(a_{i,i})_{i \geq 0}$ is an arithmetic progression of order $k + h$.

We finalize this section with a perturbation result: let x, y in a ring R such that $(y^k x^k)_{k \geq 0}$ is an arithmetic progression of order h and let $a, b \in R$ such that $a^n = 0$, $b^m = 0$, $ax = xa$ and $by = yb$; then the sequence $((y + b)^k (x + a)^k)_{k \geq 0}$ is an arithmetic progression of order $n + m + h - 2$.

Section 4. In this section we work with numerical arithmetic progressions. We recall some results about recursive equations and prove that if $(a_{cn})_{n \geq 0}$ and $(a_{dn})_{n \geq 0}$ are arithmetic progressions, then it results that $(a_{en})_{n \geq 0}$ is also an arithmetic progression, where e is the greatest common divisor of c and d .

Moreover we obtain that every arithmetic progression of positive real numbers is eventually increasing; that is, $a_n \leq a_{n+1}$, for n large enough.

The main results of this section are referred to powers of positive sequences. For example, if $a = (a_n)_{n \geq 0}$ is a positive sequence, $q, r > 0$ real numbers, $k, h \geq 0$ integer numbers, a^q is an arithmetic progression of

order k and a^r is an arithmetic progression of order h , then $rk = hq$. From this result we obtain the next classification. Given a positive sequence a , then it verifies exactly one of the following assertions:

- (1) a^q is not an arithmetic progression of order h , for all real $q > 0$ and all integer $h \geq 0$.
- (2) a is constant; hence a^q is an arithmetic progression of order h , for all real $q > 0$ and all integer $h \geq 0$.
- (3) There are unique integer $\ell \geq 1$ and real $s > 0$ such that, for every $k = 1, 2, 3, \dots$, the sequence a^{ks} is an arithmetic progression of strict order $k\ell$; moreover, if a^q is an arithmetic progression of strict order h , then $q = ks$ and $h = k\ell$, for some $k = 1, 2, 3, \dots$

Section 5. In the last section we apply the results obtained in Sections 3 and 4 to (m, q) -isometries.

Let E be a metric space, d its distance, $T : E \rightarrow E$ a map, an integer $m \geq 1$ and a real $q > 0$. Then T is an (m, q) -isometry if and only if, for all $x, y \in E$, the sequence $(d(T^n x, T^n y)^q)_{n \geq 0}$ is an arithmetic progression of order less or equal to $m - 1$.

Any power of an (m, q) -isometry is also an (m, q) -isometry. It is possible to say more: if T^c is an (m, q) -isometry and T^d is an (ℓ, q) -isometry, then T^e is an (h, q) -isometry, where e is the greatest common divisor of c and d , and h is the minimum of m and ℓ .

Using a result about arithmetic progressions allow us to prove that if $S, T : E \rightarrow E$ are maps on a metric space that commute, T is an (n, q) -isometry and S is an (m, q) -isometry, then ST is an $(m+n-1, q)$ -isometry.

An operator $T : H \rightarrow H$ on a Hilbert space H is an m -isometry if it is an $(m, 2)$ -isometry. We apply the perturbation result for arithmetic progressions on a ring to prove that if $T \in L(H)$ is an m -isometry and $Q \in L(H)$ is n -nilpotent operator such that $TQ = QT$, then $T + Q$ is an $(2n + m - 2)$ -isometry.

Finally we consider n -invertible operators: it is said that the operator S is a *left n -inverse* of the operator T if

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} S^k T^k = 0 ;$$

if that equation holds, it is said that T is a *right n -inverse* of S . This is equivalent to say that the sequence $(S^k T^k)_{k \geq 0}$ is an arithmetic progression of order $n - 1$ in $L(H)$.

Mainly, the results which we present are taken or inspired from the paper of P. Hoffmann, M. Mackey and M. O. Searcoid [19], and from [8], [9], [10], [11], [15], [12], [13] and [14].

We have included proofs in order to make the material as self contained as possible.

2. SEQUENCES IN GROUPS

In this section G denotes a commutative group and we denote additively its operation.

2.1. The difference operator.

Let $a = (a_n)_{n \geq 0}$ be a sequence in G . The difference sequence Da of a is defined by

$$Da = (a_1 - a_0, a_2 - a_1, a_3 - a_2, \dots).$$

The powers of the difference operator D are defined in the following way: $D^0 a := a$ and, for $h = 1, 2, 3, \dots$,

$$D^h a := (D^h a_n)_{n \geq 0} \quad \text{where} \quad D^h a_n := D^{h-1} a_{n+1} - D^{h-1} a_n.$$

For example, $D^2 a = (a_2 - 2a_1 + a_0, a_3 - 2a_2 + a_1, a_4 - 2a_3 + a_2, \dots)$.

Theorem 2.1. *Let a be a sequence in G . Then, for $h, n = 0, 1, 2, 3, \dots$,*

$$D^h a_n = \sum_{k=0}^h (-1)^{h-k} \binom{h}{k} a_{n+k}. \quad (2.1)$$

Proof. It is clear that Equality (2.1) is true for $h = 0$ and $h = 1$. Assume that it is valid for $h - 1$ and we will prove that is also valid for h . We have that

$$\begin{aligned} D^h a_n &= D^{h-1} a_{n+1} - D^{h-1} a_n \\ &= \sum_{k=0}^{h-1} (-1)^{h-1-k} \binom{h-1}{k} a_{n+1+k} - \sum_{k=0}^{h-1} (-1)^{h-1-k} \binom{h-1}{k} a_{n+k} \\ &= \sum_{k=0}^{h-2} (-1)^{h-1-k} \binom{h-1}{k} a_{n+1+k} + a_{n+h} + (-1)^h a_n + \sum_{k=1}^{h-1} (-1)^{h-k} \binom{h-1}{k} a_{n+k} \\ &= (-1)^h a_n + \sum_{k=1}^{h-1} (-1)^{h-k} \left(\binom{h-1}{k-1} + \binom{h-1}{k} \right) a_{n+k} + a_{n+h} \\ &= \sum_{k=0}^h (-1)^{h-k} \binom{h}{k} a_{n+k}. \end{aligned}$$

So the proof is finished. □

2.2. Some combinatorial results.

We apply the next combinatorial results in the following. The statements and their proofs are in previous papers, but we have included for sake of completeness.

Notice that for integers $n, k \geq 0$,

$$\binom{n}{k} = \frac{n(n-1) \cdots (n-k+1)}{k!},$$

being $\binom{n}{k} = 0$ if $n < k$.

Lemma 2.2. [10, Lemma 3.1] *Let $(x_j)_{j \geq 0}$ be a sequence in a group G , and let $(e_k)_{k \geq 0}$ be a sequence of integers and $(c_{k,j})_{k,j \geq 0}$ be a double sequence of integers. Then*

$$\sum_{k=0}^n e_k \sum_{j=0}^k c_{k,j} x_j = \sum_{j=0}^n x_j \sum_{k=j}^n c_{k,j} e_k .$$

for any $n = 0, 1, 2, \dots$

Proof. Note that both expressions are equal to

$$\sum_{0 \leq j \leq k \leq n} e_k c_{k,j} x_j ,$$

so the proof is completed. □

Lemma 2.3. [10, Lemma 3.2] *Let i, j be integers with $0 \leq j < i$. Then*

$$\sum_{h=0}^j (-1)^h \binom{i}{h} = (-1)^j \binom{i-1}{j} . \quad (2.2)$$

Proof. By induction on i :

(1) Case $i = j + 1$. We have that

$$\sum_{h=0}^j (-1)^h \binom{j+1}{h} = -(-1)^{j+1} \binom{j+1}{j+1} = (-1)^j ,$$

so (2.2) holds.

(2) Assume that (2.2) is true for certain $i \geq j + 1$ and we prove that it is also true for $i + 1$. Indeed,

$$\sum_{h=0}^j (-1)^h \binom{i+1}{h} = \binom{i+1}{0} - \left[\binom{i}{0} + \binom{i}{1} \right] + \dots + (-1)^j \left[\binom{i}{j-1} + \binom{i}{j} \right] = (-1)^j \binom{i}{j} .$$

This finishes the proof. □

Lemma 2.4. [10, Lemma 3.3] *Let n, h and k be integers such that $0 \leq k \leq h < n$. Then*

$$\sum_{j=k}^h (-1)^{j-k} \binom{n}{j} \binom{j}{k} = (-1)^{h-k} \frac{n(n-1) \cdots \overbrace{(n-k)} \cdots (n-h)}{k!(h-k)!} , \quad (2.3)$$

where $\overbrace{(n-k)}$ denotes that the factor $(n-k)$ is omitted.

Proof. Notice that

$$\begin{aligned}
A &:= \sum_{j=k}^h (-1)^{j-k} \binom{n}{j} \binom{j}{k} \\
&= \sum_{j=k}^h (-1)^{j-k} \frac{n(n-1) \cdots (n-k+1)(n-k)!}{k!(n-j)!(j-k)!} \\
&= \frac{n(n-1) \cdots (n-k+1)}{k!} \sum_{j=0}^{h-k} (-1)^j \binom{n-k}{j}.
\end{aligned}$$

Applying (2.2) we obtain

$$A = \frac{n(n-1) \cdots (n-k+1)}{k!} (-1)^{h-k} \binom{n-k-1}{h-k}.$$

On the other hand,

$$\begin{aligned}
B &:= (-1)^{h-k} \frac{n(n-1) \cdots \overbrace{(n-k)} \cdots (n-h)}{k!(h-k)!} \\
&= (-1)^{h-k} \frac{n(n-1) \cdots (n-k+1)(n-k-1)!}{k!(h-k)!(n-h-1)!} \\
&= (-1)^{h-k} \frac{n(n-1) \cdots (n-k+1)}{k!} \binom{n-k-1}{h-k}.
\end{aligned}$$

So we have the equality $A = B$, which proves the statement. \square

Lemma 2.5. [12, Lemma 3.6] *Let n, h be non-negative integers. Then*

$$\sum_{k=0}^h (-1)^{h-k} \frac{n(n-1) \cdots \overbrace{(n-k)} \cdots (n-h)}{k!(h-k)!} = 1. \quad (2.4)$$

Proof. Denote by $R(n, h)$ the left hand side of (2.4). Note that $R(n, h)$ is a polynomial in n of degree at most h . Moreover, for $k = 0, 1, \dots, h$, we have that

$$R(k, h) = (-1)^{h-k} \frac{k \cdots 1 \overbrace{0} \cdots (-1) \cdots (-h+k)}{k!(h-k)!} = 1.$$

Hence $R(k, h) = 1$ for $h+1$ different values, so $R(n, h) = 1$. \square

Remark 2.6. Note that

$$\begin{aligned}
\frac{n(n-1) \cdots \overbrace{(n-k)} \cdots (n-h)}{k!(h-k)!} &= \frac{n!(n-k-1)!}{k!(n-k)!(n-h-1)!(h-k)!} \\
&= \binom{n}{k} \binom{n-k-1}{h-k}.
\end{aligned}$$

Therefore Equality (2.3) can be written in the form

$$\sum_{j=k}^h (-1)^{j-k} \binom{n}{j} \binom{j}{k} = (-1)^{h-k} \binom{n}{k} \binom{n-k-1}{h-k} . \quad (2.5)$$

Analogously, Equality (2.4) is the same that

$$\sum_{k=0}^h (-1)^{h-k} \binom{n}{k} \binom{n-k-1}{h-k} = 1 . \quad (2.6)$$

2.3. The general term of a sequence.

Now we give an expression for the general term of an arbitrary sequence in function of the difference operator.

Theorem 2.7. [10, Lemma 3.4] *Let a be a sequence in G . Then, for $n = 0, 1, 2, 3, \dots$,*

$$a_n = \sum_{k=0}^n \binom{n}{k} D^k a_0 . \quad (2.7)$$

That is,

$$a_n = \sum_{k=0}^n \binom{n}{k} \sum_{h=0}^k (-1)^{k-h} \binom{k}{h} a_h .$$

Proof. It is clear that Equality (2.7) holds for $n = 0$ and $n = 1$. Assume that (2.7) is true for $0, 1, \dots, n$ and we shall prove that it is also true for $n + 1$. By (2.1) and the induction hypothesis

$$\begin{aligned} a_{n+1} &= D^{n+1} a_0 - \sum_{k=0}^n (-1)^{n+1-k} \binom{n+1}{k} a_k \\ &= D^{n+1} a_0 - \sum_{k=0}^n (-1)^{n+1-k} \binom{n+1}{k} \sum_{j=0}^k \binom{k}{j} D^j a_0 . \end{aligned}$$

From Lemma 2.2 we obtain

$$\begin{aligned} a_{n+1} &= D^{n+1} a_0 - \sum_{j=0}^n D^j a_0 \sum_{k=j}^n (-1)^{n+1-k} \binom{n+1}{k} \binom{k}{j} \\ &= D^{n+1} a_0 - \sum_{j=0}^n \binom{n+1}{j} D^j a_0 \sum_{k=j}^n (-1)^{n+1-k} \binom{n+1-j}{k-j} \\ &= \sum_{j=0}^{n+1} \binom{n+1}{j} D^j a_0 , \end{aligned}$$

because

$$\begin{aligned}
\sum_{k=j}^n (-1)^{n+1-k} \binom{n+1-j}{k-j} &= \sum_{h=0}^{n-j} (-1)^{n-j+1-h} \binom{n-j+1}{h} \\
&= (-1)^{n-j+1} \sum_{h=0}^{n-j} (-1)^h \binom{n-j+1}{h} \\
&= (-1)^{n-j+1} (-1)^{n-j} \binom{n-j}{n-j} \\
&= -1
\end{aligned}$$

by Lemma 2.3. □

3. ARITHMETIC PROGRESSIONS

In this section G denotes a commutative additive group.

Now we give the central notion of this paper.

Definition 3.1. *Let h be a non negative integer. A sequence $a = (a_n)_{n \geq 0}$ in G is called an arithmetic progression of order h if $D^h a$ is constant, so $D^{h+1} a = 0$.*

3.1. The term general of an arithmetic progression.

In the next result we give some expressions for the general term of an arithmetic progression.

Theorem 3.2. *Let a be a sequence in G . The following assertions are equivalent:*

- (1) *a is an arithmetic progression of order h .*
- (2) *For all $n \geq 0$,*

$$a_n = \sum_{k=0}^h \binom{n}{k} D^k a_0. \quad (3.8)$$

- (3) *([9, Theorem 2.1], [13, Proposition 2.2]) For all $n \geq 0$,*

$$a_n = \sum_{k=0}^h (-1)^{h-k} \frac{n(n-1) \cdots \overbrace{(n-k)}^{h-k} \cdots (n-h)}{k!(h-k)!} a_k = \sum_{k=0}^h (-1)^{h-k} \binom{n}{k} \binom{n-k-1}{h-k} a_k. \quad (3.9)$$

- (4) *[19, Proposition 3.7] For all $n \geq 0$,*

$$a_n = \frac{\sum_{k=0}^h (-1)^{h-k} \binom{h}{k} \frac{1}{n-k} a_k}{\sum_{k=0}^h (-1)^{h-k} \binom{h}{k} \frac{1}{n-k}}. \quad (3.10)$$

(5) [18, Lemma 2.2] *There is a polynomial p_a , with coefficients in G of degree less or equal to h , such that $p_a(n) = a_n$, for any $n \geq 0$; that is, there are $\gamma_h, \gamma_{h-1}, \dots, \gamma_1, \gamma_0$ in G such that*

$$a_n = \gamma_h n^h + \gamma_{h-1} n^{h-1} + \dots + \gamma_2 n^2 + \gamma_1 n + \gamma_0, \quad (3.11)$$

where $\gamma_h = \frac{1}{h!} D^h a_0$ and $\gamma_0 = a_0$.

Proof. (1) \implies (2) By definition, $D^k a_0 = 0$, for any $k \geq h+1$. Applying (2.7) we obtain the expression of (2).

(2) \implies (3) By (2.1) and (3.9) we have

$$\begin{aligned} a_n &= \sum_{k=0}^h \binom{n}{k} D^k a_0 \\ &= \sum_{k=0}^h \binom{n}{k} \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} a_i \\ &= \sum_{k=0}^h \sum_{i=0}^k (-1)^{k-i} \binom{n}{k} \binom{k}{i} a_i. \end{aligned}$$

Applying Lemmas 2.2 and 2.4,

$$\begin{aligned} a_n &= \sum_{i=0}^h a_i \sum_{k=i}^h (-1)^{k-i} \binom{n}{k} \binom{k}{i} \\ &= \sum_{i=0}^h (-1)^{h-i} \frac{n(n-1) \cdots \overbrace{(n-i)}^{h-i} \cdots (n-h)}{i!(h-i)!} a_i. \end{aligned}$$

(3) \iff (4) It is enough to prove that

$$\frac{n(n-1) \cdots \overbrace{(n-k)}^{h-k} \cdots (n-h)}{k!(h-k)!} = \frac{\binom{h}{k} \frac{1}{n-k}}{\sum_{i=0}^h (-1)^{h-i} \binom{h}{i} \frac{1}{n-i}};$$

equivalently,

$$\begin{aligned} h! &= n(n-1) \cdots (n-h) \sum_{i=0}^h (-1)^{h-i} \binom{h}{i} \frac{1}{n-i} \\ &= h! \sum_{i=0}^h (-1)^{h-i} \frac{n(n-1) \cdots \overbrace{(n-i)}^{h-i} \cdots (n-h)}{i!(h-i)!}. \end{aligned}$$

This is true by Lemma 2.5.

(3) \implies (5) Note that

$$\frac{n(n-1) \cdots \overbrace{(n-i)}^{h-i} \cdots (n-h)}{k!(h-k)!}$$

is a polynomial in n of degree h , hence a_n is a polynomial $p_a(n)$ in n of degree less or equal to h ; consequently

$$a_n = \gamma_h n^h + \gamma_{h-1} n^{h-1} + \cdots + \gamma_2 n^2 + \gamma_1 n + \gamma_0 ,$$

for some $\gamma_h, \gamma_{h-1}, \dots, \gamma_1, \gamma_0$ in G .

It is clear that $a_0 = p_a(0) = \gamma_0$. From (3.9) we obtain that the coefficient of n^h is

$$\begin{aligned} \sum_{k=0}^h (-1)^{h-k} \frac{1}{k!(h-k)!} a_k &= \frac{1}{h!} \sum_{k=0}^h (-1)^{h-k} \binom{h}{k} a_k \\ &= \frac{1}{h!} D^h a_0 . \end{aligned}$$

(5) \implies (1) If a_n is given by (3.11), we have

$$\begin{aligned} Da_n &= a_{n+1} - a_n \\ &= \sum_{k=0}^h \gamma_k (n+1)^k - \sum_{k=0}^h \gamma_k n^k \\ &= \sum_{k=0}^h \gamma_k \sum_{i=0}^{k-1} \binom{k}{i} n^i \\ &= \sum_{k=1}^h \eta_{k-1} n^{k-1} \\ &= \sum_{k=0}^{h-1} \eta_k n^k , \end{aligned}$$

for certain η_i in G ($i = 0, \dots, h-1$). So, $D^h a_n$ is a polynomial in n of degree less or equal to 1. Therefore $D^{h+1} a = 0$, which proves the statement. \square

In view of the above theorem, the arithmetic progressions are also called *polynomial sequences*.

Remark 3.3. In the proof of Proposition 3.7 in [19], the authors observe that the polynomial p_a associated to the arithmetic progression a of order h is the Lagrange polynomial of degree less or equal to h which interpolates $(0, a_0), (1, a_1), \dots, (h, a_h)$. Moreover, by the normal form of the Lagrange polynomial it is obtained (3.9), using the Newton form yields (3.8) and using the barycenter form yields (3.10).

Definition 3.4. A sequence $a = (a_n)_{n \geq 0}$ in G is called an arithmetic progression of strict order h if it is an arithmetic progression of order h with $h = 0$ or if it is an arithmetic progression of order $h \geq 1$, but is not of order $h-1$.

Theorem 3.5. Let a be a sequence in G . The following assertions are equivalent:

(1) a is an arithmetic progression of strict order h .

(2) For all $n \geq 0$,

$$a_n = \sum_{k=0}^h \binom{n}{k} D^k a_0 \quad \text{and} \quad D^h a_0 \neq 0. \quad (3.12)$$

(3) For all $n \geq 0$,

$$a_n = \sum_{k=0}^h (-1)^{h-k} \frac{n(n-1) \cdots \overbrace{(n-k)}^h \cdots (n-h)}{k!(h-k)!} a_k = \sum_{k=0}^h (-1)^{h-k} \binom{n}{k} \binom{n-k-1}{h-k} a_k \quad \text{and} \quad D^h a_0 \neq 0. \quad (3.13)$$

(4) For all $n \geq 0$,

$$a_n = \frac{\sum_{k=0}^h (-1)^{h-k} \binom{h}{k} \frac{1}{n-k} a_k}{\sum_{k=0}^h (-1)^{h-k} \binom{h}{k} \frac{1}{n-k}} \quad \text{and} \quad D^h a_0 \neq 0. \quad (3.14)$$

(5) There is a polynomial p_a , with coefficients in G of degree exactly h , such that $p_a(n) = a_n$, for any $n \geq 0$; that is, there are $\gamma_h \neq 0, \gamma_{h-1}, \dots, \gamma_1, \gamma_0$ in G such that

$$a_n = \gamma_h n^h + \gamma_{h-1} n^{h-1} + \cdots + \gamma_2 n^2 + \gamma_1 n + \gamma_0, \quad (3.15)$$

being $\gamma_h = \frac{1}{h!} D^h a_0 \neq 0$ and $\gamma_0 = a_0$.

Proof. Taking into account Theorem 3.2 it is enough to prove that (1) \implies (5) because the other implications are clear. If the polynomial p_a has degree less than h , then the sequence a is an arithmetic progression of order $h-1$, hence it is not an arithmetic progression of strict order h . \square

3.2. Subsequences of an arithmetic progression.

Let $a = (a_n)_{n \geq 0}$ be an arithmetic progressions of strict order $h \geq 0$. Two simple results, but very useful, are the following:

- (1) Fixed an integer $k \geq 1$, the subsequence $(a_{n+k})_{n \geq 0}$ is also an arithmetic progression of strict order h .
- (2) The sequence $A = (A_n)_{n \geq 0}$, defined by

$$A_n = a_0 + a_1 + \cdots + a_{n-1} + a_n,$$

is an arithmetic progression of strict order $h+1$ since $DA = (a_{n+1})_{n \geq 0}$ is an arithmetic progression of strict order h .

Let $b = (b_n)_{n \geq 0}$ be a subsequence of the sequence $a = (a_n)_{n \geq 0}$; that is, there exists a strictly increasing sequence $g = (g_n)_{n \geq 0}$ of nonnegative integers such that

$$b_n = a_{g_n} .$$

The subsequence b of a is characterized by the sequence of steps $s = (s_n)_{n \geq 0}$:

$$s_0 = g_0 \quad , \quad s_n = g_n - g_{n-1} \quad (n \geq 1) .$$

Notice that

$$g_n = s_0 + s_1 + \cdots + s_{n-1} + s_n . \quad (3.16)$$

Proposition 3.6. *Let $a = (a_n)_{n \geq 0}$ be an arithmetic progression of strict order $h \geq 0$ and let $b = (b_n)_{n \geq 0}$ be a subsequence of a with sequence of steps $s = (s_n)_{n \geq 0}$, which is an arithmetic progression of strict order k . Then b is an arithmetic progression of strict order $h(k+1)$.*

Proof. Let p_a be the polynomial associated to a which has an expression as (3.15). As s is an arithmetic progression of strict order k , we have that the sequence g given by (3.16) is an arithmetic progressions of strict order $k+1$, hence its associated polynomial p_g have degree $k+1$:

$$g_n = p_g(n) = \sigma_{k+1}n^{k+1} + \sigma_k n^k + \cdots + \sigma_1 n + \sigma_0 \quad (\sigma_{k+1} \neq 0) .$$

Consequently

$$\begin{aligned} b_n &= a_{g_n} \\ &= p_a(g_n) = \alpha_h g_n^h + \alpha_{h-1} g_n^{h-1} + \cdots + \alpha_1 g_n + \alpha_0 \\ &= \alpha_h p_g(n)^h + \alpha_{h-1} p_g(n)^{h-1} + \cdots + \alpha_1 p_g(n) + \alpha_0 \\ &= \alpha_h \sigma_{k+1}^h n^{h(k+1)} + \cdots , \end{aligned}$$

which is a polynomial in n of degree $h(k+1)$. Therefore b is an arithmetic progression of strict order $h(k+1)$. \square

Corollary 3.7. *Let $a = (a_n)_{n \geq 0}$ be an arithmetic progressions of strict order $h \geq 0$ and let $b = (b_n)_{n \geq 0}$ be the subsequence of a defined by*

$$b_n = a_{dn} \quad (n \geq 0) ,$$

for certain integer $d \geq 1$. Then b is an arithmetic progression of strict order h .

Proof. The result is obtained from the above proposition taking as sequence of steps $s = (s_n)_{n \geq 0}$ the constant sequence $s = (d)_{n \geq 0}$, which is an arithmetic progression of order 0. \square

3.3. Sequences of arithmetic progressions.

Now we work with a double sequence $(a_{i,j})_{i,j \geq 0}$ such that the files $(a_{i,j})_{j \geq 0}$ and the columns $(a_{i,j})_{i \geq 0}$ are arithmetic progressions. We obtain that the diagonal sequence $(a_{i,i})_{i \geq 0}$ is also an arithmetic progression. The results of this subsection are basically in [13] and [18].

First we give a necessary lemma.

Lemma 3.8. *Let $(a_i)_{i \geq 0}$ be an arithmetic progression of order h and let n be a positive integer with $n > 1$.*

Then

$$\sum_{i=0}^{h+n} \binom{h+n}{i} (-1)^{h+n+1-i} i(i-1) \cdots (i-\ell) a_i \quad (3.17)$$

is zero, for any $\ell \in \{0, 1, \dots, n-2\}$.

Proof. Fixed $\ell \in \{0, 1, \dots, n-2\}$. The equation (3.17) is equivalent to

$$\sum_{i=\ell+1}^{h+n} \binom{h+n}{i} (-1)^{h+n+1-i} i(i-1) \cdots (i-\ell) a_i. \quad (3.18)$$

Note that

$$\begin{aligned} \binom{h+n}{i} i(i-1) \cdots (i-\ell) &= \frac{(h+n)!}{i!(h+n-i)!} i(i-1) \cdots (i-\ell) \\ &= \frac{(h+n) \cdots (h+n-\ell)(h+n-\ell-1)!}{(i-\ell-1)!(h+n-i)!} \\ &= \binom{h+n-\ell-1}{i-\ell-1} \prod_{j=1}^{\ell+1} (h+n-j+1). \end{aligned}$$

Then (3.18) agrees with

$$\begin{aligned} &\prod_{j=1}^{\ell+1} (h+n-j+1) \sum_{i=\ell+1}^{h+n} \binom{h+n-\ell-1}{i-\ell-1} (-1)^{h+n-i+1} a_i \\ &= \prod_{j=1}^{\ell+1} (h+n-j+1) \sum_{i=0}^{h+n-\ell-1} \binom{h+n-\ell-1}{i} (-1)^{h+n-\ell-i} a_{\ell+1+i}. \end{aligned} \quad (3.19)$$

As $(a_i)_{i \geq 0}$ is an arithmetic progression of order h , we have that $(a_i)_{i \geq 0}$ is an arithmetic progression of order $h + n - \ell - 2$ for all $\ell \in \{0, 1, \dots, n - 2\}$; that is,

$$\sum_{i=0}^{h+n-\ell-1} (-1)^{h+n-\ell-1-i} \binom{h+n-\ell-1}{i} a_i = 0.$$

Hence we obtain the result. \square

Theorem 3.9. [18, Corollary 2.5]. *Let $(a_{i,j})_{i,j \geq 0}$ be a double sequence such that the sequence $(a_{i,j})_{j \geq 0}$ is an arithmetic progression of order $k \geq 0$, for all $i \geq 0$, and the sequence $(a_{i,j})_{i \geq 0}$ is an arithmetic progression of order $h \geq 0$, for all $j \geq 0$. Then the diagonal sequence $(a_{i,i})_{i \geq 0}$ is an arithmetic progression of order $k + h$.*

Proof. For $k = 0$ is clear. Assume that $k > 0$. Denote

$$A := \sum_{r=0}^{k+h+1} \binom{k+h+1}{r} (-1)^{k+h+1-r} a_{r,r}.$$

Let us prove that $A = 0$. From part (3) of Theorem 3.2 we obtain

$$\begin{aligned} A &= \sum_{r=0}^{k+h+1} \binom{k+h+1}{r} (-1)^{k+h+1-r} \left(\sum_{i=0}^k g(r,i,k) a_{r,i} \right) \\ &= \sum_{i=0}^k \sum_{r=0}^{k+h+1} \binom{k+h+1}{r} (-1)^{k+h+1-r} g(r,i,k) a_{r,i}, \end{aligned}$$

where

$$g(r,i,k) := (-1)^{k-i} \frac{r(r-1) \cdots \overbrace{(r-i)}^{k-i} \cdots (r-k)}{i!(k-i)!}.$$

By elementary properties of polynomials we have that there exist scalars $(\eta_{i,j})_{j=0}^k$ such that

$$r(r-1) \cdots \overbrace{(r-i)}^{k-i} \cdots (r-k) = \eta_{i,0} + \eta_{i,1}r + \eta_{i,2}r(r-1) + \cdots + \eta_{i,k}r(r-1) \cdots (r-k+1) \quad (3.20)$$

for $i \in \{0, 1, \dots, k\}$. By equality (3.20) and Lemma 3.8 we obtain the result. \square

3.4. Perturbation of arithmetic progressions by nilpotents on a ring.

In this subsection the setting is a ring R with unity e , whose operations are denoted by $+$ and \cdot , being \cdot not necessarily commutative. The main result is included in [15].

Given an integer $n \geq 1$, an element $a \in R$ is called n -nilpotent if $a^n = 0$ but $a^{n-1} \neq 0$.

In the next theorem we consider a sequence $(y^k x^k)_{k \geq 0}$ in R which is an arithmetic progression of strict order h . By part (5) of Theorem 3.5, for every $k \geq 0$,

$$y^k x^k = \sum_{\ell=0}^h c_\ell k^\ell \quad (c_h \neq 0), \quad (3.21)$$

where $c_\ell \in R$ ($0 \leq \ell \leq h$) do not depend on k .

Theorem 3.10. *Let R be a ring. Let $x, y \in R$ such that $(y^k x^k)_{k \geq 0}$ is an arithmetic progression of strict order h . Let $a, b \in R$ such that a is n -nilpotent, b is m -nilpotent, $ax = xa$ and $by = yb$. Then the sequence $((y + b)^k (x + a)^k)_{k \geq 0}$ is an arithmetic progression of order $n + m + h - 2$. Moreover, it is of strict order $n + m + h - 2$ whenever $b^{m-1} y^{n-m} c_h a^{n-1} \neq 0$, if $m \leq n$, or whenever $b^{m-1} c_h x^{m-n} a^{n-1} \neq 0$, if $m > n$.*

Proof. Denote by $c \wedge d$ the minimum of c and d . We have, for every $k \geq 0$,

$$(y + b)^k (x + a)^k = \left(\sum_{i=0}^k \binom{k}{i} b^i y^{k-i} \right) \left(\sum_{j=0}^k \binom{k}{j} x^{k-j} a^j \right).$$

Consequently, for every $k \geq 0$,

$$(y + b)^k (x + a)^k = \sum_{i=0}^{k \wedge (m-1)} \sum_{j=0}^{k \wedge (n-1)} \binom{k}{i} \binom{k}{j} b^i y^{k-i} x^{k-j} a^j. \quad (3.22)$$

Consider two cases.

Case 1. $m \leq n$. Taking into account (3.21), then (3.22) can be written in the following way:

$$\begin{aligned} (y + b)^k (x + a)^k &= \sum_{i=0}^{k \wedge (m-1)} \sum_{j=i}^{k \wedge (n-1)} \binom{k}{i} \binom{k}{j} b^i y^{j-i} y^{k-j} x^{k-j} a^j \\ &+ \sum_{i=1}^{k \wedge (m-1)} \sum_{j=0}^{i-1} \binom{k}{i} \binom{k}{j} b^i y^{k-i} x^{k-i} x^{i-j} a^j \\ &= \sum_{i=0}^{k \wedge (m-1)} \sum_{j=i}^{k \wedge (n-1)} \binom{k}{i} \binom{k}{j} b^i y^{j-i} \left(\sum_{\ell=0}^h c_\ell (k-j)^\ell \right) a^j \\ &+ \sum_{i=1}^{k \wedge (m-1)} \sum_{j=0}^{i-1} \binom{k}{i} \binom{k}{j} b^i \left(\sum_{\ell=0}^h c_\ell (k-i)^\ell \right) x^{i-j} a^j \\ &= \sum_{i=0}^{k \wedge (m-1)} \sum_{j=i}^{k \wedge (n-1)} \sum_{\ell=0}^h \binom{k}{i} \binom{k}{j} (k-j)^\ell b^i y^{j-i} c_\ell a^j \\ &+ \sum_{i=1}^{k \wedge (m-1)} \sum_{j=0}^{i-1} \sum_{\ell=0}^h \binom{k}{i} \binom{k}{j} (k-i)^\ell b^i c_\ell x^{i-j} a^j. \end{aligned}$$

We write:

$$p_{i,j,\ell}(k) := \binom{k}{i} \binom{k}{j} (k-j)^\ell, \quad u_{i,j,\ell} := b^i y^{j-i} c_\ell a^j,$$

being $0 \leq i \leq k \wedge (m-1)$, $i \leq j \leq k \wedge (n-1)$ and $0 \leq \ell \leq h$; analogously,

$$q_{i,j,\ell}(k) := \binom{k}{i} \binom{k}{j} (k-i)^\ell, \quad v_{i,j,\ell} := b^i c_\ell x^{i-j} a^j,$$

for $0 \leq j < i \leq k \wedge (m-1)$ and $0 \leq \ell \leq h$. Hence

$$(y+b)^k(x+a)^k = \sum_{i=0}^{k \wedge (m-1)} \sum_{i \leq j}^{k \wedge (n-1)} \sum_{\ell=0}^h p_{i,j,\ell}(k) u_{i,j,\ell} + \sum_{i=1}^{k \wedge (m-1)} \sum_{j=0}^{i-1} \sum_{\ell=0}^h q_{i,j,\ell}(k) v_{i,j,\ell}.$$

Note that the polynomial $p_{i,j,\ell}(k)$ has degree $i+j+\ell$, which is maximum with value $m+n+h-2$; analogously, the degree of $q_{i,j,\ell}(k)$ is $i+j+\ell$, which is maximum with value $m+m+h-3$. Therefore $(y+b)^k(x+a)^k$ is a polynomial of degree less or equal to $m+n+h-2$. Actually the degree is $m+n+h-2$ whenever the coefficient of $k^{m+n+h-2}$ is non-null; that is,

$$u_{m-1,n-1,h} = b^{m-1} y^{n-m} c_h a^{n-1} \neq 0.$$

Case 2. $m > n$. Analogously to Case 1, (3.22) can be written

$$\begin{aligned} (y+b)^k(x+a)^k &= \sum_{i=0}^{k \wedge (n-1)} \sum_{j=i}^{k \wedge (n-1)} \binom{k}{i} \binom{k}{j} b^i y^{j-i} y^{k-j} x^{k-j} a^j \\ &+ \sum_{j=0}^{k \wedge (n-1)} \sum_{i=j+1}^{k \wedge (m-1)} \binom{k}{i} \binom{k}{j} b^i y^{k-i} x^{k-i} x^{i-j} a^j \\ &= \sum_{i=0}^{k \wedge (n-1)} \sum_{j=i}^{k \wedge (n-1)} \sum_{\ell=0}^h \binom{k}{i} \binom{k}{j} (k-j)^\ell b^i y^{j-i} c_\ell a^j \\ &+ \sum_{j=0}^{k \wedge (n-1)} \sum_{i=j+1}^{k \wedge (m-1)} \sum_{\ell=0}^h \binom{k}{i} \binom{k}{j} (k-i)^\ell b^i c_\ell x^{i-j} a^j. \end{aligned}$$

We write:

$$p_{i,j,\ell}(k) := \binom{k}{i} \binom{k}{j} (k-j)^\ell, \quad u_{i,j,\ell} := b^i y^{j-i} c_\ell a^j,$$

being $0 \leq i \leq j \leq k \wedge (n-1)$ and $0 \leq \ell \leq h$; similarly,

$$q_{i,j,\ell}(k) := \binom{k}{i} \binom{k}{j} (k-i)^\ell, \quad v_{i,j,\ell} := b^i c_\ell x^{i-j} a^j,$$

being $0 \leq j \leq k \wedge (n-1)$, $j+1 \leq i \leq k \wedge (m-1)$ and $0 \leq \ell \leq h$. Then

$$(y+b)^k(x+a)^k = \sum_{i=0}^{k \wedge (n-1)} \sum_{j=i}^{k \wedge (n-1)} \sum_{\ell=0}^h p_{i,j,\ell}(k) u_{i,j,\ell} + \sum_{j=0}^{k \wedge (n-1)} \sum_{i=j+1}^{k \wedge (m-1)} \sum_{\ell=0}^h q_{i,j,\ell}(k) v_{i,j,\ell}.$$

The degree of $p_{i,j,\ell}(k)$ is $i+j+\ell$, which is maximum with value $n+n+h-2$, and the degree of $q_{i,j,\ell}(k)$ is $i+j+\ell$, with maximum $m+n+h-2$. Hence $(y+b)^k(x+a)^k$ is a polynomial of degree less or equal to $m+n+h-2$. Really the degree is $m+n+h-2$ if the coefficient of $k^{m+n+h-2}$ is non-null; that is,

$$v_{m-1,n-1,h} = b^{m-1} c_h x^{n-m} a^{n-1} \neq 0.$$

Thus the proof is finished. \square

Corollary 3.11. *Let R be a ring. Let $x \in R$ such that $(x^k)_{k \geq 0}$ is an arithmetic progression of strict order h and let $a \in R$ be an n -nilpotent such that $ax = xa$. Then the sequence $((x + a)^k)_{k \geq 0}$ is an arithmetic progression of order $n + h - 1$; moreover, it is of strict order $n + h - 1$ if $a^{n-1}c_h x^{n-1} \neq 0$, where*

$$x^k = \sum_{\ell=0}^h c_\ell k^\ell .$$

4. NUMERICAL ARITHMETIC PROGRESSIONS

In this section we consider sequences of scalars, mainly sequences of positive numbers.

Throughout this section h, k, ℓ are integers and $q > 0$ a real number.

4.1. Recursive equations.

The content of this subsection is taken from [8, p. 251]; see also [1, p. 104] and [20, Theorem 3.7].

The sequence $(a_n)_{n \geq 0}$ of scalars is an arithmetic progression of strict order h if and only if, for every $n = 0, 1, 2, \dots$,

$$D^{h+1}a_n = \sum_{k=0}^{h+1} (-1)^{h+1-k} \binom{h+1}{k} a_{n+k} = 0 \quad \text{and} \quad D^h a_n \neq 0 . \quad (4.23)$$

Notice that (4.23) is a recursive equation. Let us introduce some classical results to solve this type of equations. We are interested in sequences $(y_n)_{n \geq 0}$ which verify the recursive equation

$$y_{h+1+n} + \gamma_h y_{h+n} + \gamma_{h-1} y_{h-1+n} + \dots + \gamma_1 y_{1+n} + \gamma_0 y_n = 0 , \quad (4.24)$$

for certain $h \geq 0$ and any $n \geq 0$, being γ_i complex numbers ($0 \leq i \leq h$). The *characteristic polynomial* of the equation (4.24) is given by

$$q(z) = z^{h+1} + \gamma_h z^h + \gamma_{h-1} z^{h-1} + \dots + \gamma_1 z + \gamma_0 ,$$

which can be written in the form

$$q(z) = (z - z_1)^{h_1} (z - z_2)^{h_2} \dots (z - z_k)^{h_k} , \quad (4.25)$$

where $h_1 + h_2 + \dots + h_k = h + 1$ and $z_i \neq z_g$ for $i \neq g$. It is well known (see for example [20, Theorem 3.7] and [1, page 104]) that the set of all complex sequences which verify (4.24) is a vectorial subspace of the

space $\mathbb{C}^{\mathbb{N}}$ of all complex sequences, it has dimension $h + 1$ and a basis is formed by the sequences

$$\begin{aligned}
& (z_1^n)_{n \geq 0}, (nz_1^n)_{n \geq 0}, (n^2 z_1^n)_{n \geq 0}, \dots, (n^{h_1-1} z_1^n)_{n \geq 0}, \\
& (z_2^n)_{n \geq 0}, (nz_2^n)_{n \geq 0}, (n^2 z_2^n)_{n \geq 0}, \dots, (n^{h_2-1} z_2^n)_{n \geq 0}, \\
& \dots\dots\dots, \\
& (z_j^n)_{n \geq 0}, (nz_j^n)_{n \geq 0}, (n^2 z_j^n)_{n \geq 0}, \dots, (n^{h_j-1} z_j^n)_{n \geq 0}.
\end{aligned} \tag{4.26}$$

Therefore there exists an identification between the recursive equation (4.24), the characteristic polynomial (4.25), the subspace of sequences which satisfy the recursive equation and its basis (4.26).

Consider now an arithmetic progression $a = (a_n)_{n \geq 0}$ of order h which verifies (4.23), for every $n \geq 0$. The characteristic polynomial associated is $q_1(z) = (z-1)^{h+1}$, hence the sequence $(a_n)_{n \geq 0}$ is a linear combination of the sequences $(1)_{n \geq 0}, (n)_{n \geq 0}, (n^2)_{n \geq 0}, \dots, (n^h)_{n \geq 0}$. This is another way to obtain the expression (3.11). Note that the sequence a has strict order h if a is not a linear combination of only the sequences $(1)_{n \geq 0}, (n)_{n \geq 0}, (n^2)_{n \geq 0}, \dots, (n^{h-1})_{n \geq 0}$.

4.2. Subsequences of numerical sequences.

In the next result we consider a sequence $(a_n)_{n \geq 0}$ such that its subsequences $(a_{cn})_{n \geq 0}$ and $(a_{dn})_{n \geq 0}$ are arithmetic progressions. This result is essentially [8, Theorem 3.6].

Theorem 4.1. *Let $a = (a_n)_{n \geq 0}$ be a numerical sequence. Suppose that $(a_{cn})_{n \geq 0}$ is an arithmetic progression of strict order h and $(a_{dn})_{n \geq 0}$ is an arithmetic progressions of strict order $k \geq 0$, for $c, d \geq 1$ and $h, k \geq 0$. Then $(a_{en})_{n \geq 0}$ is an arithmetic progression of strict order ℓ , being e the greatest common divisor of c and d , and ℓ the minimum of h and k .*

Proof. As $(a_{cn})_{n \geq 0}$ is an arithmetic progression of strict order h satisfies the recursive equation

$$\sum_{i=0}^{h+1} (-1)^{h+1-i} \binom{h+1}{i} a_{c(i+j)} = 0, \tag{4.27}$$

for all $j \geq 0$. The equation (4.27) has the characteristic polynomial $p_1(z) := q_c(z)^{h+1} := (z^c - 1)^{h+1}$. We call V_c to the subspace of $\mathbb{C}^{\mathbb{N}}$ formed by all complex sequences which verify (4.27). Then V_c has dimension $\dim V_c = (h+1)c$.

Analogously, $(a_{dn})_{n \geq 0}$ satisfies the equation

$$\sum_{i=0}^{k+1} (-1)^{k+1-i} \binom{k+1}{i} a_{d(i+j)} = 0, \tag{4.28}$$

for all $j \geq 0$. The characteristic polynomial of (4.28) is $p_2(z) := q_d(z)^{k+1} = (z^d - 1)^{k+1}$. Now the set of all the sequences which verify (4.28) is a vectorial subspace V_d of $\mathbb{C}^{\mathbb{N}}$ and $\dim V_d = d(k+1)$.

Therefore the sequences which verify both equations (4.27) and (4.28) are the sequences in the subspace $V_c \cap V_d$, those characteristic polynomial is the greatest common divisor of $q_c(z)^{h+1}$ and $q_d(z)^{k+1}$, which is the polynomial $q_e(z)^{\ell+1} := (z^e - 1)^{\ell+1}$, where e is the greatest common divisor of c and d , and ℓ is the minimum of h and k . Consequently $(a_{en})_{n \geq 0}$ is an arithmetic progression of strict order ℓ . \square

4.3. Monotony of positive arithmetic progressions.

We say that the sequence $a = (a_n)_{n \geq 0}$ of real numbers is *positive* if $a_n > 0$ for any $n \geq 0$. This subsection is dedicated to positive arithmetic progressions.

Proposition 4.2. *Let a be an arithmetic progression of order h of positive real numbers. Then the sequence a is eventually increasing; that is, there is a positive integer n_0 such that, for any integer $n \geq n_0$, we have that $a_n \leq a_{n+1}$. Moreover, if a is not constant, then*

$$\lim_{n \rightarrow \infty} a_n = \infty .$$

Proof. If a is constant the result is clear. Assume that a is not constant, so is an arithmetic progression of strict order $h \geq 1$. The sequence $(a_n)_{n \geq 0}$ satisfies (3.11) with positive leader coefficient γ_h , so $a_n \rightarrow \infty$ as $n \rightarrow \infty$ and it is eventually increasing. \square

Example 4.3. The sequence $a = (a_n)_{n \geq 0}$ of natural numbers, given by $a_n = n^2 - 2n + 2$ ($n \geq 0$) is a positive arithmetic progression of strict order 2. The first terms of a are 2, 1, 2, 5, 10..., hence a is not monotone.

Example 4.4. The sequence of natural numbers $a = (a_n)_{n \geq 0} = (n^3 - 5n^2 + 8n + 1)_{n \geq 0}$ is a positive arithmetic progression of strict order 3, is increasing, but is not strictly increasing because $a_1 = a_2 = 5$: $a = (1, 5, 5, 7, 17 \dots)$.

4.4. Powers of positive sequences.

Given a positive sequence $a = (a_n)_{n \geq 0}$ and a real number $q > 0$, we define the sequence $a^q := (a_n^q)_{n \geq 0}$.

Proposition 4.5. *Let $a = (a_n)_{n \geq 0}$ be a positive sequence, $q, r > 0$ and $k, h \geq 0$. If a^q is an arithmetic progression of strict order k and if a^r is an arithmetic progression of strict order h , then a^{r+q} is an arithmetic progression of strict order $k + h$.*

Proof. The polynomials associated to a^q and a^r are, respectively,

$$a_n^q = \alpha_k n^k + \alpha_{k-1} n^{k-1} + \cdots + \alpha_0 \quad , \quad a_n^r = \beta_h n^h + \beta_{h-1} n^{h-1} + \cdots + \beta_0 \quad (\alpha_k \neq 0, \beta_h \neq 0) .$$

Then

$$a_n^{r+q} = \alpha_k \beta_h n^{k+h} + \cdots + \alpha_0 \beta_0 \quad (\alpha_k \beta_h \neq 0) .$$

Therefore a^{r+q} is an arithmetic progression of strict order $k+h$. □

Proposition 4.6. [19, Proposition 4.1] *Let $a = (a_n)_{n \geq 0}$ be a positive sequence, $q, r > 0$ and $k, h \geq 0$. If a^q is an arithmetic progression of strict order k and if a^r is an arithmetic progression of strict order h , then*

$$rk = hq . \tag{4.29}$$

Proof. Consider the polynomials associated to the sequences a^q and a^r , respectively: for all integers $n \geq 0$,

$$a_n^q = \alpha_k n^k + \alpha_{k-1} n^{k-1} + \cdots + \alpha_1 n + \alpha_0 \quad (\alpha_k \neq 0) ,$$

$$a_n^r = \beta_h n^h + \beta_{h-1} n^{h-1} + \cdots + \beta_1 n + \beta_0 \quad (\beta_h \neq 0) .$$

Note that $k = 0 \iff h = 0$, hence (4.29) holds. Assume $k, h \geq 1$. In this case there exist

$$\lim_{n \rightarrow \infty} \frac{a_n^q}{n^k} = \alpha_k > 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{a_n^r}{n^h} = \beta_h > 0 .$$

Thus we have the equality

$$\lim_{n \rightarrow \infty} \frac{a_n^{qr}}{n^{rk}} = \lim_{n \rightarrow \infty} \frac{a_n^{rq}}{n^{hq}} ,$$

so $kr = hq$. □

Taking $k, h \geq 1$ in the statement of Proposition 4.6, hence a is non-constant, it is possible to improve the result.

Proposition 4.7. [19, Lemma 4.2] *Let $a = (a_n)_{n \geq 0}$ be a non-constant positive sequence, $q, r > 0$ and $k, h \geq 1$. If a^q is an arithmetic progression of strict order k and if a^r is an arithmetic progression of strict order h , then a^t is an arithmetic progression of strict order d , where $d = \gcd\{k, h\}$ and*

$$t = \frac{qd}{k} = \frac{rd}{h} . \tag{4.30}$$

Proof. Let $P := p_{a^q}$ and $Q := p_{a^r}$ the polynomials associated to the sequences a^q and a^r , respectively. For any integer $n \geq 0$,

$$P(n) = a_n^q = \alpha_k n^k + \alpha_{k-1} n^{k-1} + \cdots + \alpha_1 n + \alpha_0 \quad (\alpha_k \neq 0),$$

$$Q(n) = a_n^r = \beta_h n^h + \beta_{h-1} n^{h-1} + \cdots + \beta_1 n + \beta_0 \quad (\beta_h \neq 0).$$

From Proposition 4.6 we obtain $rk = hq$. Hence

$$P(n)^h = a_n^{qh} = a_n^{kr} = Q(n)^k.$$

Therefore $P^h = Q^k$, so P and Q have the same zeroes. If z_i ($1 \leq i \leq \ell$) is a common zero with multiplicity u_i in P and multiplicity v_i in Q , then $u_i h = v_i k$. Moreover, if $d = \gcd\{k, h\}$, then we obtain that k/d is divisor of u_i . Thus, for certain integers w_i ($1 \leq i \leq \ell$), for all $n \geq 0$,

$$P(n) = (n - z_1)^{u_1} \cdots (n - z_\ell)^{u_\ell} = ((n - z_1)^{w_1} \cdots (n - z_\ell)^{w_\ell})^{k/d} = R(n)^{k/d},$$

where R is a polynomial of degree d . Consequently

$$R(n) = P(n)^{d/k} = a_n^{qd/k} = a_n^t,$$

being $t = qd/k$. Hence a^t is an arithmetic progression of strict order d . As $rk = hq$, we can write $t = rd/h$. \square

Proposition 4.8. [19, Proposition 4.3] *Let $a = (a_n)_{n \geq 0}$ be a positive sequence. Then a satisfies exactly one of the following assertions:*

- (1) a^q is not an arithmetic progression of order h , for all $q > 0$ and $h \geq 0$.
- (2) a is constant; hence a^q is an arithmetic progression of order h , for all $q > 0$ and $h \geq 0$.
- (3) There are unique integer $\ell \geq 1$ and real $s > 0$ such that, for every $k = 1, 2, 3, \dots$, the sequence a^{ks} is an arithmetic progression of strict order $k\ell$. Moreover, if a^q is an arithmetic progression of strict order h , then $q = ks$ and $h = k\ell$, for some $k = 1, 2, 3, \dots$

Proof. Suppose that (1) and (2) fail, hence a is not constant. Let $\ell \geq 1$ the least integer such that a^s is an arithmetic progression of strict order ℓ for some $s > 0$. From Proposition 4.6 we obtain that s is unique. Let $P := p_{a^s}$ be the polynomial of degree ℓ associated to the sequence a^s . Then, for every $k \geq 1$, the polynomial P^k has degree $k\ell$ and

$$P(n)^k = a_n^{ks},$$

for $n \geq 0$. Thus a^{ks} is an arithmetic progression of strict order $k\ell$. Conversely, if a^q is an arithmetic progression of strict order $h \geq 1$, then a^t is an arithmetic progression of strict order d , where $d = \gcd\{\ell, h\}$

and $t = \frac{qd}{h} = \frac{sd}{\ell}$. From the minimality of ℓ we have that $d = \ell$ and then there is $k = \frac{q}{s} = \frac{h}{\ell} = 1, 2, 3, \dots$ such that $h = k\ell$ and $q = ks$. \square

The Proposition 4.8 leads to following notion.

Definition 4.9. Let $s, q > 0$ be real numbers and $\ell, h > 0$ be integers. Given a non-constant positive sequence a , we say that a is an arithmetic progression of proper order (s, ℓ) whenever a^s is an arithmetic progression of strict order ℓ and if a^q is an arithmetic progression of strict order h , then $s \leq q$ and $\ell \leq h$.

Notice that, if some power a^q of a positive sequence a is an arithmetic progression, then there exist a real $s > 0$ and an integer $\ell \geq 1$ such that a is an arithmetic progression of proper order (s, ℓ) . Moreover a^{ks} is of strict order $k\ell$ and of order $h \geq k\ell$, for all integer $k \geq 1$ and any integer h .

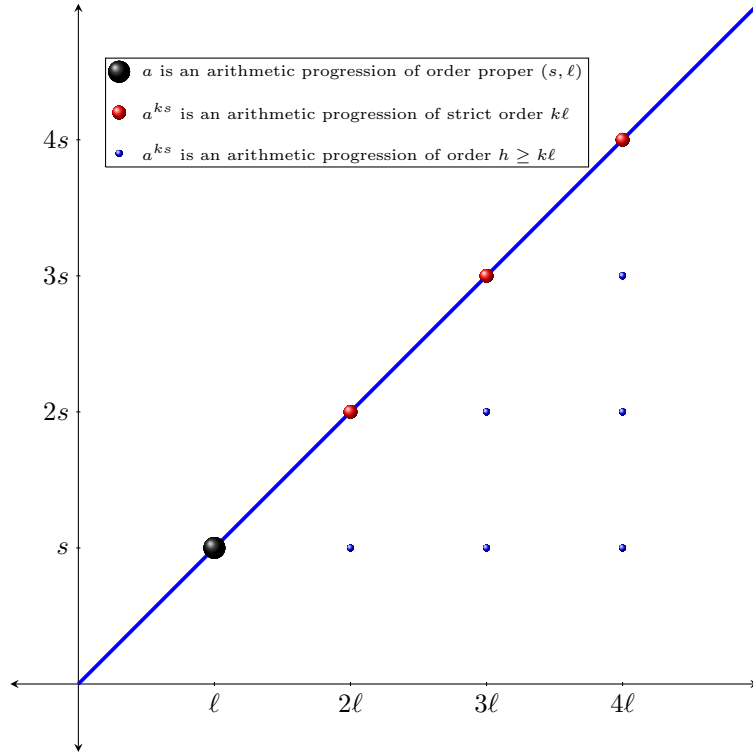


FIGURE 1. Graphical interpretation of the order of a non-constant arithmetic progression

Remark 4.10. For a positive sequence a , in [19] the authors consider the sets

$$\pi(a) := \{(h, q) : h \geq 0 \text{ } q > 0 \text{ } a^q \text{ is an arithmetic progression of order } h\},$$

$\widehat{\pi}(a) := \{(h, q) : h \geq 0, q > 0, a^q \text{ is an arithmetic progression of strict order } h\}$.

Proposition 4.8 can be formulated in the following form: the positive sequence a verifies exactly one of the following:

- (1) $\pi(a) = \widehat{\pi}(a) = \emptyset$.
- (2) $\widehat{\pi}(a) = \{0\} \times]0, \infty[$ and $\pi(a) = \{0, 1, 2, 3, \dots\} \times]0, \infty[$
- (3) there are unique integer $\ell \geq 1$ and real $s > 0$ such that

$$\widehat{\pi}(a) = \{(k\ell, ks) : k = 1, 2, 3, \dots\} \text{ and } \pi(a) = \{(h, ks) : k = 1, 2, 3, \dots; h \geq k\ell\}.$$

5. APPLICATIONS TO (m, q) -ISOMETRIES

Now we apply the results of the previous sections to (m, q) -isometries. Consider three settings: metric spaces, Banach spaces and Hilbert spaces.

In this section q, r, s denote real numbers and m, n, h, k, ℓ are integers.

5.1. (m, q) -isometries on metric spaces.

Throughout this section, E denotes a metric space, d its distance and $T : E \longrightarrow E$ a map.

Definition 5.1. A map $T : E \longrightarrow E$ is called an (m, q) -isometry if, for all $x, y \in E$,

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} d(T^k x, T^k y)^q = 0. \quad (5.31)$$

It is clear that $(1, q)$ -isometries are isometries.

Proposition 5.2. Let $T : E \longrightarrow E$ be a map. Fixed $m \geq 1$ and $q > 0$, the following assertions are equivalent:

- (1) T is an (m, q) -isometry.
- (2) For all $x, y \in E$, the sequence $(d(T^n x, T^n y)^q)_{n \geq 0}$ is an arithmetic progression of order $m - 1$.

Proof. We have that T is an (m, q) -isometry if and only if Equation (5.31) holds for all $x, y \in E$. Fix $x, y \in E$. For every integer $\ell \geq 0$,

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} d(T^k T^\ell x, T^k T^\ell y)^q = 0,$$

so

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} d(T^{k+\ell} x, T^{k+\ell} y)^q = 0. \quad (5.32)$$

Therefore $(d(T^n x, T^n y)^q)_{n \geq 0}$ is an arithmetic progression of order $m - 1$. \square

From Propositions 4.2 and 5.2 we obtain that if T is an (m, q) -isometry, then the sequence $(d(T^n x, T^n y)^q)_{n \geq 0}$ is eventually increasing.

The (m, q) -isometries are $(m + \ell, q)$ -isometries, for every $\ell \geq 0$, since the arithmetic progressions of order h are also arithmetic progressions of order $h + \ell$, for all $\ell \geq 0$. For this it is natural the following definition:

Definition 5.3. *A map $T : E \longrightarrow E$ is called a strict (m, q) -isometry if it is an (m, q) -isometry with $m = 1$ or if it is an (m, q) -isometry, but is not an $(m - 1, q)$ -isometry, for $m > 1$.*

Now we give an analogous to Proposition 5.2 for strict (m, q) -isometries. We omit the proof since is clear.

Proposition 5.4. *Let $T : E \longrightarrow E$ be a map. Fixed $m \geq 1$ and $q > 0$, the following assertions are equivalent:*

- (1) *T is a strict (m, q) -isometry.*
- (2) *For all $x, y \in E$, the sequence $(d(T^n x, T^n y)^q)_{n \geq 0}$ is an arithmetic progression of order $m - 1$ and it is an arithmetic progression of strict order $m - 1$ for some $x, y \in E$.*

By Proposition 4.7, if T is an (m, q) -isometry and an (n, r) -isometry, then

$$(m - 1)r = (n - 1)q .$$

Applying Proposition 4.8 we obtain the following result.

Proposition 5.5. *Let $T : E \longrightarrow E$ be a map. Then T satisfies exactly one of the following assertions:*

- (1) *T is not an (m, q) -isometry for all $m \geq 1$ and $q > 0$.*
- (2) *T is an isometry.*
- (3) *There are unique $m \geq 2$ and $q > 0$ such that T is a strict $((m - 1)k + 1, qk)$ -isometry, for every $k = 1, 2, 3, \dots$. Moreover, if T is an strict (n, r) -isometry, then $n = (m - 1)k + 1$ and $r = qk$ for some $k = 1, 2, 3, \dots$*

Proof. Note that if T is an strict (m, q) -isometry, then $(d(T^k x, T^k y)^q)_{k \geq 0}$ is an arithmetic progression of order $m - 1$, for all $(x, y) \in E \times E$, and $(d(T^k u, T^k v)^q)_{k \geq 0}$ is an arithmetic progression of strict order $m - 1$, for some $(u, v) \in E \times E$. Hence, for all $k \geq 1$, $(d(T^k x, T^k y)^q)_{k \geq 0}$ is an arithmetic progression of order $k(m - 1)$, for all $(x, y) \in E \times E$, and $(d(T^k u, T^k v)^q)_{k \geq 0}$ is an arithmetic progression of strict order $k(m - 1)$, for some $(u, v) \in E \times E$. Consequently T is a strict $(k(m - 1) + 1, kq)$ -isometry. \square

The above proposition and Definition 4.9 motivate the following definition.

Definition 5.6. An (m, q) -isometry $T : E \longrightarrow E$ is said to be a proper (m, q) -isometry if $m \leq n$ and $q \leq r$ whenever T is a (n, r) -isometry.

Note that any proper (m, q) -isometry is a strict (m, q) -isometry. By [19, Corollary 4.6], also it is a strict $(k(m-1)+1, kq)$ -isometry, for all positive integer k .

5.2. Distance associated to an (m, q) -isometry on a metric space.

The (m, q) -isometries become isometries for an adequate distance. The following results are analogous to [7].

Proposition 5.7. [10, Proposition 5.1] Let T be an (m, q) -isometry. For $x, y \in E$ define

$$\rho_T(x, y) := \left(\sum_{k=0}^{m-1} (-1)^{m-1-k} \binom{m-1}{k} d(T^k x, T^k y)^q \right)^{1/q}.$$

Then ρ_T is a semi-distance and moreover,

$$\rho_T(x, y)^q = (m-1)! \lim_{n \rightarrow \infty} \frac{d(T^n x, T^n y)^q}{n^{m-1}}. \quad (5.33)$$

Proof. As $(d(T^n x, T^n y)_{n \geq 0})$ is an arithmetic progression of order $m-1$, by Theorem 3.5 we can write

$$\begin{aligned} d(T^n x, T^n y)^q &= \gamma_{m-1} n^{m-1} + \gamma_{m-2} n^{m-2} + \cdots + \gamma_0 \\ &= \sum_{k=0}^{m-1} \binom{n}{k} D^k (d(T^0 x, T^0 y)^q), \end{aligned}$$

where

$$\gamma_{m-1} = \frac{1}{(m-1)!} D^{m-1} (d(T^0 x, T^0 y)^q) = \frac{1}{(m-1)!} \left(\sum_{k=0}^{m-1} (-1)^{m-1-k} \binom{m-1}{k} d(T^k x, T^k y)^q \right).$$

Therefore

$$\begin{aligned} \rho_T(x, y)^q &= (m-1)! \gamma_{m-1} \\ &= (m-1)! \lim_{n \rightarrow \infty} \frac{d(T^n x, T^n y)^q}{n^{m-1}}. \end{aligned}$$

We will show that ρ_T is a semi-metric. By (5.33) it is clear that $\rho_T \geq 0$, $\rho_T(x, x) = 0$ and $\rho_T(x, y) = \rho_T(y, x)$ for all $x, y \in E$. It remains to show the triangular inequality. Let $x, y, z \in E$. Then

$$\begin{aligned} \rho_T(x, y) &= [(m-1)!]^{1/q} \lim_{n \rightarrow \infty} \frac{d(T^n x, T^n y)}{n^{\frac{m-1}{q}}} \\ &\leq [(m-1)!]^{1/q} \lim_{n \rightarrow \infty} \frac{d(T^n x, T^n z)}{n^{\frac{m-1}{q}}} + [(m-1)!]^{1/q} \lim_{n \rightarrow \infty} \frac{d(T^n z, T^n y)}{n^{\frac{m-1}{q}}} \\ &= \rho_T(x, z) + \rho_T(z, y). \end{aligned}$$

So the proof is finished. \square

By (5.33), if T is an (m, q) -isometry, then $\rho_T(x, y) = \rho_T(Tx, Ty)$; that is, $T : (E, \rho_T) \rightarrow (E, \rho_T)$ is an isometry.

5.3. Powers and products of (m, q) -isometries on metric spaces.

Now we apply the results of subsections 2.5 and 3.2 to (m, q) -isometries.

Theorem 5.8. [10, Theorems 3.15 and 3.16], [8, Theorems 3.1 and 3.6]. *Let $T : E \rightarrow E$ be a map.*

- (1) *If T is a strict (m, q) -isometry, then any power T^k is also a strict (m, q) -isometry.*
- (2) *If T^c is a strict (m, q) -isometry and T^d is a strict (ℓ, q) -isometry, then T^e is an (h, q) -isometry, where e is the greatest common divisor of c and d , and h is the minimum of m and ℓ .*

Proof. It is useful the following notation: we define the map $T \times_q T : E \times E \rightarrow \mathbb{R}$ by $(T \times_q T)(x, y) := d(Tx, Ty)^q$.

(1) Assume that T is a strict (m, q) -isometry. Note that this is equivalent to that the sequence $(T^n \times_q T^n)_{n \geq 0}$ is an arithmetic progression of strict order $m-1$ in $\mathbb{R}^{E \times E}$. Given an integer $k \geq 1$, $(T^{kn} \times_q T^{kn})_{n \geq 0}$ is also an arithmetic progression of strict order $m-1$ by Corollary 3.7. Hence T^k is a strict (m, q) -isometry.

(2) As T^c is a strict (m, q) -isometry we have that $(T^{ck} \times_q T^{ck})_{k \geq 0}$ is an arithmetic progression of strict order $m-1$. Analogously, from T^d a strict (ℓ, q) -isometry we obtain that $(T^{dk} \times_q T^{dk})_{k \geq 0}$ is an arithmetic progression of strict order $\ell-1$. Applying Theorem 4.1 it results that $(T^{ek} \times_q T^{ek})_{k \geq 0}$ is an arithmetic progression of strict order $h-1$, so T^e is an (h, q) -isometry, being $e = \gcd(m, \ell)$ and h the minimum of m and ℓ . \square

Corollary 5.9. ([10, Corollary 3.17], [8, Corollary 3.7]). *Let E be a metric space and $T : E \rightarrow E$ be a map. For positive integers h, n, m and real number $q \geq 0$,*

- (1) *If T is an (m, q) -isometry and T^h is an isometry, then T is an isometry.*

- (2) If T^h and T^{h+1} are (m, q) -isometries, then T is an (m, q) -isometry.
- (3) If T^h is an (m, q) -isometry and T^{h+1} is an (n, q) -isometry with $m < n$, then T is an (m, q) -isometry.

Theorem 5.10. ([10, Theorem 3.14], [8, Theorem 3.3]). *Let E be a metric space, $S, T : E \rightarrow E$ maps such that commute, q positive real number and n, m positive integers. If T is an (n, q) -isometry and S is an (m, q) -isometry, then ST is an $(m + n - 1, q)$ -isometry.*

Proof. Fix $x, y \in E$. Consider the sequence $(a_{i,j})_{i,j \geq 0}$ defined by $a_{i,j} := d(S^i T^j x, S^i T^j y)^q$. As S is an (m, q) -isometry, the sequence $(a_{i,j})_{i \geq 0}$ is an arithmetic progression of order $m - 1$, for all $j \geq 0$. Taking into account that $TS = ST$ and that T is (n, q) -isometry, the sequence $(a_{i,j})_{j \geq 0}$ is an arithmetic progression of order $n - 1$, for all $i \geq 0$. Applying Theorem 3.9 we have that $(a_{i,i})_{i \geq 0}$ is an arithmetic progression of order $m + n - 2$. Therefore ST is an $(m + n - 1, q)$ -isometry. \square

5.4. (m, q) -isometries on Banach spaces.

Throughout this subsection, X denotes a Banach space and $\|\cdot\|$ its norm, and $T : X \rightarrow X$ a linear map.

The notion of (m, q) -isometry can be adapted to the setting of Banach space in the following way:

Proposition 5.11. *A linear map $T : X \rightarrow X$ is an (m, q) -isometry if and only if, for all $x \in X$,*

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \|T^k x\|^q = 0. \quad (5.34)$$

Proof. It is enough note that $d(x, y) = \|x - y\|$ and $\|T^k x - T^k y\| = \|T^k(x - y)\|$. \square

Proposition 5.12. *Let $T : X \rightarrow X$ be a linear map. Fixed $m \geq 1$ and $q > 0$, the following assertions are equivalent:*

- (1) T is an (m, q) -isometry.
- (2) For all $x \in X$, the sequence $(\|T^n x\|^q)_{n \geq 0}$ is an arithmetic progression of order $m - 1$.

Now we give an analogous to Proposition 5.2 for strict (m, q) -isometries. We omit the proof since is clear.

Proposition 5.13. *Let $T : X \rightarrow X$ be a linear map. Fixed $m \geq 1$ and $q > 0$, the following assertions are equivalent:*

- (1) T is a strict (m, q) -isometry.
- (2) For all $x \in X$, the sequence $(\|T^n x\|^q)_{n \geq 0}$ is an arithmetic progression of order $m - 1$ and it is an arithmetic progression of strict order $m - 1$ for some $x \in X$.

5.5. m -isometries on Hilbert spaces.

Throughout this subsection, H denotes a Hilbert space and $\langle \cdot \rangle$ its inner product, $T : H \longrightarrow H$ a (linear bounded) operator and T^* its adjoint.

As Hilbert spaces are considered Banach spaces, we can apply the results given in the previous section. However, the case $q = 2$ can be expressed in a special way. By this, we give the following definition:

Definition 5.14. *An operator $T : H \longrightarrow H$ is an m -isometry if it is an $(m, 2)$ -isometry.*

Proposition 5.15. *Let $T \in L(H)$. The following statements are equivalent:*

- (1) *T is an m -isometry; that is, for every $x \in H$, the sequence $(\|T^k x\|^2)_{k \geq 0}$ is an arithmetic progression of order $m - 1$ in \mathbb{R} ; that is for all $\ell \geq 0$,*

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \|T^{k+\ell} x\|^2 = 0 .$$

Equivalently, for every $x \in H$,

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \|T^k x\|^2 = 0 . \quad (5.35)$$

- (2) *The following operator equality*

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} T^{*k} T^k = 0 \quad (5.36)$$

*holds. Equivalently, the sequence $(T^{*k} T^k)_{k \geq 0}$ is an arithmetic progression of order $m - 1$ in $L(H)$; that is, for all $\ell \geq 0$,*

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} T^{*k+\ell} T^{k+\ell} = 0 .$$

- (3) *For every $x \in H$, the sequence $(T^{*k} T^k x)_{k \geq 0}$ is an arithmetic progression of order $m - 1$ in H ; that is, for all $\ell \geq 0$,*

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} T^{*k+\ell} T^{k+\ell} x = 0 .$$

Equivalently, for every $x \in H$,

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} T^{*k} T^k x = 0 .$$

- (4) *For every $x, y \in H$, the sequence $(\langle T^{*k} T^k x, y \rangle)_{k \geq 0}$ is an arithmetic progression of order $m - 1$ in \mathbb{C} ; that is, for all $\ell \geq 0$,*

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \langle T^{*k+\ell} T^{k+\ell} x, y \rangle = 0 .$$

Equivalently, for every $x, y \in H$,

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \langle T^{*k} T^k x, y \rangle = 0 .$$

Proof. Note that, for every $x \in H$,

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \|T^k x\|^2 = \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \langle T^k x, T^k x \rangle = \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \langle T^{*k} T^k x, x \rangle = 0 .$$

Taking into account that the operator $\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} T^{*k} T^k$ is self adjoint, we obtain that the conditions (1) and (2) are equivalent. The other implications are clear. \square

5.6. A perturbation result on Hilbert spaces.

Now apply Theorem 3.10, about perturbation of arithmetic progressions by nilpotents on rings, to m -isometries.

We use the following notation:

$$\beta_{m-1}(T) := \sum_{i=0}^{m-1} (-1)^{m-1-i} \binom{m-1}{i} T^{*i} T^i .$$

Theorem 5.16. [11]. *Let $T \in L(H)$ be a strict m -isometry. Let $Q \in L(H)$ be an n -nilpotent operator such that $TQ = QT$. Then $T + Q$ is an $(2n + m - 2)$ -isometry. Moreover, is a strict $(2n + m - 2)$ -isometry if and only if $Q^{*n-1} \beta_{m-1}(T) Q^{n-1} \neq 0$.*

Proof. We apply Theorem 3.10 taking $y = T^*$, $x = T$, $b = Q^*$ and $a = Q$, being both a and b n -nilpotent. Then $((T + Q)^{*k} (T + Q)^k)_{k \geq 0}$ is an arithmetic progression of order $n + n + m - 3$. Consequently $T + Q$ is a $(2n + m - 2)$ -isometry. Moreover, $T + Q$ is a strict $(2n + m - 2)$ -isometry whenever $b^{n-1} c_{m-1} a^{n-1} \neq 0$; that is

$$Q^{*n-1} \beta_{m-1}(T) Q^{n-1} = Q^{*n-1} \left(\sum_{i=0}^{m-1} (-1)^{m-1-i} \binom{m-1}{i} T^{*i} T^i \right) Q^{n-1} \neq 0 ,$$

since $c_{m-1} = \frac{1}{(m-1)!} \beta_{m-1}(T)$ by Theorem 3.5. Hence the proof is finished. \square

5.7. n -invertible operators on Hilbert spaces.

Suppose that $S, T : H \rightarrow H$ are operators on the Hilbert space H . We say that S is a *left n -inverse* of T if

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} S^k T^k = 0 . \quad (5.37)$$

If equation (5.37) holds, it is said that T is a *right n -inverse* of S ; equivalently $(S^k T^k)_{k \geq 0}$ is an arithmetic progression of order $n - 1$ in the algebra $L(H)$. We say that S is an *left strict n -inverse* of T if the sequence $(S^k T^k)_{k \geq 0}$ is an arithmetic progression of strict order $n - 1$ in $L(H)$; in this case we also say that T is an *right strict n -inverse* of S . In [18] and [22] are studied the right and left n -inverses. They are related with m -isometries. Note that T is an m -isometry whenever T^* is a left m -inverse of T ; that is, T is an right m -inverse of T^* . Many results about m -isometries can be extended to m -inverses. We give a perturbation result similar to Theorem 5.16.

We denote

$$\beta_{n-1}(S, T) := \sum_{i=0}^{n-1} (-1)^{m-1-i} \binom{m-1}{i} S^i T^i.$$

In [15] is proved the following result.

Theorem 5.17. *Let $S \in L(H)$ be a left strict n -inverse of T . Let $P \in L(H)$ be an h -nilpotent and $Q \in L(H)$ be a k -nilpotent such that $SP = PS$ and $TQ = QT$. Then $S + P$ is a left $(n + h + k - 2)$ -inverse of $T + Q$. Moreover, it is a strict left $(n + h + k - 2)$ -inverse if and only if $P^{h-1} S^{h-k} \beta_{n-1}(S, T) Q^{k-1} \neq 0$, if $k \leq h$, or whenever $P^{h-1} \beta_{n-1}(S, T) T^{k-h} Q^{k-1} \neq 0$, if $h \leq k$.*

Proof. As S is a left strict n -inverse of T , we have that $(S^k T^k)_{k \geq 0}$ is an arithmetic progression of strict order $n - 1$. In Theorem 3.10 we take $y = S$, $x = T$, $b = P$ and $a = Q$. Then the sequence $((y + b)^k (x + a)^k)_{k \geq 0} = ((S + P)^k (T + Q)^k)_{k \geq 0}$ is an arithmetic progression of order $n + h + k - 3$. Therefore $S + P$ is a left $(n + h + k - 2)$ -inverse of $T + Q$. Moreover it is strict whenever

$$b^{h-1} y^{h-k} c_{n-1} a^{k-1} = P^{h-1} S^{h-k} \beta_{n-1}(S, T) Q^{k-1} \neq 0,$$

if $k \leq h$, or whenever

$$b^{h-1} c_{n-1} x^{k-h} a^{k-1} = P^{h-1} \beta_{n-1}(S, T) T^{k-h} Q^{k-1} \neq 0,$$

if $h \leq k$, since $c_{n-1} = \frac{1}{(n-1)!} \beta_{n-1}(S, T)$ by Theorem 3.5. □

Acknowledgements: The first author is partially supported by grant of Ministerio de Ciencia e Innovación, Spain, proyect no. MTM2011-26538.

REFERENCES

- [1] R. P. Agarwal, *Difference equations and inequalities. Theory, methods, and applications*, Monographs and Textbooks in Pure and Applied Mathematics, 155. Marcel Dekker, Inc., New York, 1992.
- [2] J. Agler, A disconjugacy theorem for Toeplitz operators. *Amer. J. Math.* 112 (1) (1990) 1-14.

- [3] J. Agler, M. Stankus, m -isometric transformations of Hilbert space. I. *Integral Equations Operator Theory*, 21 (4) (1995) 383-429.
- [4] J. Agler, M. Stankus, m -isometric transformations of Hilbert space. II. *Integral Equations Operator Theory*, 23 (1) (1995) 1-48.
- [5] J. Agler, M. Stankus, m -isometric transformations of Hilbert space. III. *Integral Equations Operator Theory*, 24 (4) (1995) 379-421.
- [6] J. Alonso, Arithmetic sequences of higher order. *Fibonacci Quart.*, 14 (2) (1976) 147-152.
- [7] F. Bayart, m -isometries on Banach spaces, *Math. Nachr.*, 284 (2011) 2141-2147.
- [8] T. Bermúdez, C. Díaz-Mendoza, A. Martínón, Powers of m -isometries. *Studia Math.*, 208 (2012), 249-255.
- [9] T. Bermúdez, I. Marrero, A. Martínón, On the orbit of an m -isometry. *Integral Equations Operator Theory*, 64 (2009) 487-494.
- [10] T. Bermúdez, A. Martínón, V. Müller, (m, q) -isometries on metric spaces. *J. Operator Theory*, to appear.
- [11] T. Bermúdez, A. Martínón, V. Müller, J. A. Noda, Perturbation of an m -isometry by a nilpotent. *Abstr. Appl. Anal.*, to appear.
- [12] T. Bermúdez, A. Martínón, E. Negrín, Weighted shift operators which are m -isometries. *Integral Equations Operator Theory*, 68 (2010) 301-312.
- [13] T. Bermúdez, A. Martínón, J. Noda, Products of m -isometries. *Linear Algebra Appl.*, 438 (2013) 80-86.
- [14] T. Bermúdez, A. Martínón, J. Noda, An isometry plus an nilpotent operator is an m -isometry. *J. Math. Anal. Appl.*, 407 (2013) 505-512.
- [15] T. Bermúdez, A. Martínón, J. A. Noda, On ∞ -inverses. In preparation.
- [16] F. Botelho, On the existence of n -isometries on ℓ_p spaces. *Acta Sci. Math. (Szeged)*, 76 (2010) 183192.
- [17] V. Dlab, Arithmetic progressions of higher order. *Teaching Math. Comp. Science*, 9 (2) (2011) 225-239.
- [18] B. P. Duggal, V. Müller, Tensor product of left n -invertible operators. *Studia Math.*, 215 (2) (2013) 113-125.
- [19] P. Hoffmann, M. Mackey, M. O. Searcóid, On the second parameter of an (m, p) -isometry. *Integral Equations Operator Theory*, 71 (2011) 389-405.
- [20] W. G. Kelley, A. C. Peterson, *Difference Equations. An Introduction with Applications*, Academic Press, 1991.
- [21] O. A. M. Sid Ahmed, m -isometric operators on Banach spaces. *Asian-Eur. J. Math.*, 3 (2010) 1-19.
- [22] O. A. M. Sid Ahmed, Some properties of m -isometries and m -invertible operators on Banach spaces, *Acta Math. Sci. Ser. B Engl. Ed.*, 32B (2) (2012) 520-530.

E-mail address: tbermude@ull.es, anmarce@ull.es, joannoda@gmail.com

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, UNIVERSIDAD DE LA LAGUNA, 38271 LA LAGUNA (TENERIFE), SPAIN